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Theory and Methodology

A note on deriving weights from pairwise comparison ratio matrices

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Abstract: This paper addresses the Cook and Kress method regarding the use of mathematical programming to derive weights from pairwise comparison ratio matrices. We note that the Cook and Kress formulation has the shortcoming that alternative optimal solutions may occur, which would lead to an infinite set of possible weights. This paper proposes to resolve the problem by formulating a Phase II optimization procedure to be solved using quadratic programming. This mathematical formulation is based on the determination of a weight vector that is statistically most likely to cause the associated pairwise comparison matrix. It is concluded that the Cook and Kress method accompanied with the proposed Phase II approach can always derive a unique set of weights from a pairwise comparison matrix.

Keywords: Pairwise comparison matrices; Alternatives weighting; Mathematical programming; Likelihood function

1. The Cook and Kress method

The problem of deriving weights from pairwise comparison ratio matrices is to extract a weight vector

$$w = (w_i), \quad w_i > 0, \ i = 1, \ldots, n,$$

which is usually normalized as $\Sigma_{i=1}^{n} w_i = 1$, where n is the number of compared alternatives, from a positive reciprocal matrix

$$A = (a_{ij}), \quad a_{ij} > 0, \quad a_{ii} = 1, \quad a_{ij} = 1/a_{ji}, \ i, j = 1, \ldots, n, \tag{1.1}$$

where a_{ij} is the intensity of preference of alternative i over j. Pairwise comparison matrix A is said to be *consistent* if and only if $a_{ij} \cdot a_{jk} = a_{ik}$, $i, j, k = 1, \ldots, n$. This condition is equivalent to the condition that element a_{ij} is expressed as $a_{ij} = w_i / w_j$, $i, j = 1, \ldots, n$, for some positive values w_i and w_j (Cook and Kress, 1988; Takahashi, 1990). Therefore, we can instantly obtain a unique weight vector w corresponding to the pairwise comparison matrix A when the matrix is consistent. But, in general, the given pairwise comparison matrix is inconsistent, so that we need methods for deriving weights.

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Cook and Kress (1988) proposed a unique method. They first proved that the unique distance between any two pairwise comparison matrices P and W with property (1.1) is given by

$$d(P, Q) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |\ln(p_{ij}/q_{ij})|, \tag{1.2}$$

where p_{ij} and q_{ij} are the elements of matrices P and Q, respectively. Then, the problem of deriving weight vector w from pairwise comparison matrix A can be considered as follows: Although the given pairwise comparison matrix is generally inconsistent, it must be reflecting the unknown weights. That is, the given matrix can be considered close to the consistent matrix corresponding to the unknown weights. Therefore, the problem is to find a consistent matrix as close as possible to the given matrix A.

Using distance measure (1.2) and noting that the elements of consistent pairwise comparison matrix are w_i/w_j, i, $j = 1, \ldots, n$, the problem can be expressed in mathematical programming form:

$$\begin{aligned} &\text{Minimize} && \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |\ln[a_{ij}/(w_i/w_j)]| \\ &\text{subject to} && \prod_{i=1}^{n} w_i = 1, \\ & && w_i > 0, \quad i = 1, \ldots, n, \end{aligned} \tag{1.3}$$

where $\prod_{i=1}^{n} w_i = 1$ is a normalization condition on the weight vector. Let $x_i = \ln w_i$, $i = 1, \ldots, n$, then problem (1.3) is written as follows:

$$\begin{aligned} &\text{Minimize} && \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |\ln a_{ij} - x_i + x_j| \\ &\text{subject to} && \sum_{i=1}^{n} x_i = 0, \\ & && (x_i \text{ unconstrained}, \quad i = 1, \ldots, n). \end{aligned}$$

This problem is equivalent to the following goal programming problem:

$$\begin{aligned} &\text{Minimize} && \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (d_{ij}^{+} + d_{ij}^{-}) \\ &\text{subject to} && x_i - x_j - d_{ij}^{+} + d_{ij}^{-} = \ln a_{ij}, \quad i = 1, \ldots, n-1, \quad j = i+1, \ldots, n, \\ & && \sum_{i=1}^{n} x_i = 0, \\ & && d_{ij}^{+}, d_{ij}^{-} \geqq 0, \quad i = 1, \ldots, n-1, \quad j = i+1, \ldots, n, \\ & && (x_i \text{ unconstrained}, \quad i = 1, \ldots, n), \end{aligned} \tag{1.4}$$

where d_{ij}^{+} and d_{ij}^{-} are *difference variables* in goal programming.

From the solution to problem (1.4), the weights are given by

$$w_i = \exp(x_i), \quad i = 1, \ldots, n. \tag{1.5}$$

The weights normalized additionally to sum up to unity are

$$\tilde{w}_i = w_i / \sum_{j=1}^{n} w_j, \quad i = 1, \ldots, n. \tag{1.6}$$

That is the outline of the Cook and Kress method.

In this way, the Cook and Kress method is a good method that applies the concept of distance measures appropriately. The shortcoming of this method is that there often exist plural consistent matrices at the minimum distance from the given pairwise comparison matrix. This implies that the linear programming (LP) problem (1.4) may have the minimum value at plural extreme points, which would lead to an infinite set of possible weights.

Let us show an example: Suppose that an (inconsistent) pairwise comparison matrix is given by

$$A = \begin{pmatrix} 1 & 3 & 2 \\ \frac{1}{3} & 1 & \frac{1}{2} \\ \frac{1}{2} & 2 & 1 \end{pmatrix}. \tag{1.7}$$

Then, using available LP softwares, we can easily have an optimal solution $x^1 = (0.5493\ -0.5493\ 0)$ to LP problem (1.4) and the corresponding weight vector $\tilde{w}^1 = (0.5234\ 0.1745\ 0.3022)$. However, $\tilde{w}^1$ is not a unique weight vector. In this case, at least $x^2 = (0.5014\ -0.5973\ 0.0959)$ and $x^3 = (0.6932\ -0.6932\ 0)$ are also optimal extreme-point solutions to LP problem (1.4). Moreover, since every convex combination of x^1, x^2 and x^3 is the optimal solution, an infinite number of weight vectors are obtained. That is, the weights to be got are indefinite.

Unfortunately, the case where LP problem (1.4) has plural optimal extreme-point solutions often occurs because the Cook and Kress method takes the form of, so to speak, the *least-absolutes* method. In the next section, we consider a way to get over this shortcoming of the Cook and Kress method.

2. Proposing the Phase II approach

Let us consider the problem dividing into two phases as follows: LP problem (1.4) yields a consistent matrix at the minimum distance from the given pairwise comparison matrix (Phase I). When problem (1.4) has plural optimal solutions, we must select one out of the optimal solutions (Phase II). The Cook and Kress method completes Phase I. We need a method to deal with Phase II, which should be added to the Cook and Kress method.

Let $D_{\min}$ be the minimum distance obtained in Phase I. Then, the problem in Phase II is to select one weight vector out of the weight vectors that compose consistent matrices at distance $D_{\min}$ from the given pairwise comparison matrix. For this purpose, a criterion other than distance is needed. We adopt *likelihood* as the criterion. That is, we consider, among the weight vectors composing consistent matrices at distance $D_{\min}$, which weight vector is statistically most likely to cause the given pairwise comparison matrix.

We consider on the basis of the usual multiplicative model as follows:

$$a_{ij} = (w_i/w_j)\varepsilon_{ij}, \quad i, j = 1, \ldots, n, \tag{2.1}$$

where ε_{ij} is the positive error term following a certain distribution. This model means that pairwise comparison ratio a_{ij} is caused by the ratio of unknown weights w_i/w_j and disturbance ε_{ij}. Taking logarithms of (2.1) yields

$$\ln a_{ij} = \ln w_i - \ln w_j + \ln \varepsilon_{ij}, \quad i, j = 1, \ldots, n,$$

where the mean of $\ln \varepsilon_{ij}$ is supposed to be zero.

Assuming that $\ln \varepsilon_{ij}$ follows a normal distribution with mean zero and variance σ^2, $N(0, \sigma^2)$, the probability density of a_{ij} is given by

$$f(a_{ij}) = \frac{1}{\sqrt{2\pi}\,\sigma a_{ij}} \exp\left[-\frac{(\ln a_{ij} - \ln w_i + \ln w_j)^2}{2\sigma^2} \right], \quad i, j = 1, \ldots, n.$$

When samples a_{ij}, $i, j = 1, \dots, n$, are given, the logarithmic likelihood function is

$$\ln L(w_1, \dots, w_n, \sigma^2) = -\sum_{i=1}^{n} \sum_{j=1}^{n} \ln\left(\sqrt{2\pi}\,\sigma a_{ij}\right) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\ln a_{ij} - \ln w_i + \ln w_j\right)^2.$$

To get the maximum likelihood estimates of w_i, $i = 1, \dots, n$, we can maximize this logarithmic likelihood function. Noting that the estimates of w_i do not depend on the estimate of σ^2 and that

$$\ln a_{ii} = 0, \qquad \ln a_{ij} = -\ln a_{ji}, \quad i, j = 1, \dots, n,$$

the above is equivalent to minimizing

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\ln w_i - \ln w_j - \ln a_{ij}\right)^2. \tag{2.2}$$

As for the assumption that disturbance $\ln \varepsilon_{ij}$ follows $N(0, \sigma^2)$, there is the following objection (e.g., Basak, 1991; Saaty, 1990): If $\ln \varepsilon_{ij}$ follows $N(0, \sigma^2)$, then ε_{ij} follows the log-normal distribution

$$\left[1/\left(\sqrt{2\pi}\,\sigma\varepsilon_{ij}\right)\right] \exp\left[-\left(\ln \varepsilon_{ij}\right)^2/(2\sigma^2)\right], \tag{2.3}$$

with mean $\exp(\frac{1}{2}\sigma^2)$ (> 1). Therefore, the mean of a_{ij} in model (2.1) cannot be w_i/w_j.

We do not think this is an anomaly. When the disturbance ε_{ij} follows log-normal distribution (2.3), the *median* of ε_{ij} is 1, i.e.,

$$\operatorname{Prob}\left[0 \leqq \varepsilon_{ij} \leqq 1\right] = \operatorname{Prob}\left[1 \leqq \varepsilon_{ij} \leqq \infty\right].$$

We think this is natural because it implies that, in model (2.1), a_{ij} is disturbed by ε_{ij} so that probabilities of underestimating and overestimating w_i/w_j would be the same. However, since the ranges [0, 1] and [1, ∞] are different, the mean of ε_{ij} is greater than 1. That is, for model (2.1), it is not a fallacy that the mean of a_{ij} cannot be w_i/w_j, but a natural consequence in the case where the median of a_{ij} should be w_i/w_j.

Therefore, to select one out of the weight vectors composing consistent matrices at distance $D_{\min}$ in Phase II, we can minimize expression (2.2) subject to the distance from the given pairwise comparison matrix is $D_{\min}$. Letting $x_i = \ln w_i$, $i = 1, \dots, n$, and noting that $D_{\min}$ is obtained as the minimum value of the objective function to LP problem (1.4), the problem in Phase II can be expressed in the following quadratic programming (QP) form:

$$\begin{aligned}
&\text{Minimize} && \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(x_i - x_j - \ln a_{ij}\right)^2 \\
&\text{subject to} && x_i - x_j - d_{ij}^+ + d_{ij}^- = \ln a_{ij}, \quad i = 1, \dots, n-1, \quad j = i+1, \dots, n, \\
& && \sum_{i=1}^{n} x_i = 0, \\
& && \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(d_{ij}^+ + d_{ij}^-\right) = D_{min}, \\
& && d_{ij}^+, d_{ij}^- \geqq 0, \quad i = 1, \dots, n-1, \quad j = i+1, \dots, n, \\
& && \left(x_i \text{ unconstrained}, \quad i = 1, \dots, n\right).
\end{aligned} \tag{2.4}$$

We can get a solution to problem (2.4) using mathematical programming softwares, e.g., LINDO (Schrage, 1984), and the Appendix guarantees that the solution is a unique optimal solution. Using (1.5) and (1.6), we can obtain only one weight vector.

In the case of example (1.7), LP problem (1.4) yields the minimum objective function value 0.2877. Letting $D_{\min} = 0.2877$ and solving QP problem (2.4), we obtain the optimal solution $x = (0.5973\ \ -0.5973\ \ 0)$ and can determine a unique set of weights $\tilde{w} = (0.5396\ \ 0.1634\ \ 0.2970)$.

In general, it is difficult to know if an LP problem has plural optimal solutions when we solve it using softwares. Therefore, we should deal with Phase II whether LP problem (1.4) has plural optimal solutions or not. Then, this paper proposes to take this Phase II optimization approach together with the Cook and Kress method so that we can always obtain a unique set of weights. That is, the procedure using mathematical programming to derive weights from a pairwise comparison matrix can be summarized as the following two-phase approach:

Phase I. Using the Cook and Kress method, i.e., solving LP problem (1.4), obtain the minimum distance from the given pairwise comparison matrix to consistent matrices, $D_{\min}$.

Phase II. Solving QP problem (2.4) using the value of $D_{\min}$, obtain the optimal solution and determine weight vector $\tilde{w}$.

Appendix

If the set of feasible solutions to QP problem (2.4) has only one element, problem (2.4) has only one optimal solution.

Let x^1 and x^2 be the feasible solutions to QP problem (2.4) and suppose that $x^1 \neq x^2$, then

$$\sum_{i=1}^{n} x_i^1 = \sum_{i=1}^{n} x_i^2 = 0. \tag{A.1}$$

Let

$$g(x) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (x_i - x_j - \ln a_{ij})^2$$

and consider the function

$$h(\lambda) = g\left[(1-\lambda)x^1 + \lambda x^2\right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left[x_i^1 - x_j^1 - \lambda\left(x_i^1 - x_j^1 - x_i^2 + x_j^2\right) - \ln a_{ij}\right]^2,$$

where λ is a scalar and $0 \leqq \lambda \leqq 1$. Then,

$$h''(\lambda) = 2\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(x_i^1 - x_j^1 - x_i^2 + x_j^2\right)^2 \geqq 0, \tag{A.2}$$

with equality if and only if

$$x_i^1 - x_j^1 - x_i^2 + x_j^2 = 0, \quad i = 1,\ldots,n-1, \quad j = i+1,\ldots,n.$$

This condition can be written as follows: For any i, $i = 1,\ldots,n$,

$$x_i^1 - x_i^2 = x_j^1 - x_j^2, \quad j = 1,\ldots,n.$$

Summing up with respect to j and using (A.1), we get

$$nx_i^1 - nx_i^2 = \sum_{j=1}^{n} x_j^1 - \sum_{j=1}^{n} x_j^2 = 0,$$

so that $x_i^1 = x_i^2$, $i = 1,\ldots,n$. Therefore, equality of (A.2) holds if and only if $x^1 = x^2$. This implies that $h''(\lambda) > 0$ in the feasible region to QP problem (2.4). It can be proved that $h''(\lambda) > 0$ means $g(x)$ is strictly convex (e.g., Mangasarian, 1969). Therefore, QP problem (2.4) has only one optimal solution.

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